

Fluctuation Theorems for Synchronization of Interacting Pólya's urns

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Abstract

We consider a model of N two-colors urns in which the reinforcement of each urn depends also on the content of all the other urns. This interaction is of mean-field type and it is tuned by a parameter $\alpha \in [0, 1]$; in particular, for $\alpha = 0$ the N urns behave as N independent Pólya's urns. For $\alpha > 0$ urns synchronize, in the sense that the fraction of balls of a given color converges a.s. to the same (random) limit in all urns. In this paper we study fluctuations around this synchronized regime. The scaling of these fluctuations depends on the parameter α . In particular the standard scaling $t^{-1/2}$ appears only for $\alpha > 1/2$. For $\alpha \geq 1/2$ we also determine the limit distribution of the rescaled fluctuations. We use the notion of stable convergence, which is stronger than convergence in distribution.

Keywords: Central limit theorem, Fluctuation theorem, Interacting system, Stable convergence, Synchronization, Urn model.

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1 Introduction

In this paper we continue the study of synchronization for a model of interacting Pólya's urn that has been introduced in [9]. This study is motivated by the attempt of understanding the role of reinforcement in synchronization phenomena. Here the word *synchronization* is meant in the wide sense of “coherent behavior of the majority”, that could be time stationary or time periodic. Experimental results in the context of cellular and neuronal systems have stimulated the formulation and the analysis of stylized stochastic models that could reveal the origin of such phenomena, in particular in systems that are not time-reversible (see e.g. [13, 25]). The wide majority of the models proposed consists of time-homogeneous Markov processes; with this choice, long-time correlations and *aging* are usually ruled out.

One way of breaking time-homogeneity and adding memory to the dynamics consists in introducing a reinforcement mechanism. The most popular stylized model in this context is the Pólya's urn model. In the simplest version, the model consists of an urn which contains balls of two different colors (for example, at time $t = 0$, $a \geq 1$ red and $b \geq 1$ black balls). At each discrete time a ball is drawn out and it is replaced in the urn together with another ball of the same color. Let Z_t be the fraction of red balls at time t , namely, the conditional probability of drawing a red ball at time $t + 1$, given the fraction of the red balls at time t . A

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well known result (see for instance [17] or [20]) states that $(Z_t)_t$ is a bounded martingale and in particular $\lim_{t \rightarrow \infty} Z_t = Z_\infty$ a.s., where Z_∞ has Beta distribution with parameters a and b .

In [9] an interacting version of this model is formulated. Consider a set of $N > 1$ Pólya's urns and introduce a "mean field interaction" among them:

- at time 0, each urn contains $a \geq 1$ red and $b \geq 1$ black balls;
- at each time $t + 1$, a new ball is introduced in each urn and, given the fraction $Z_t(i)$, for $1 \leq i \leq N$, of red balls in each urn i at time t , the ball added in urn j is, independently of what happens for all the other urns, red (otherwise it is black) with conditional probability $\alpha Z_t + (1 - \alpha) Z_t(j)$, where Z_t is the total fraction of red balls in the system at time t , i.e. $Z_t = \frac{1}{N} \sum_{i=1}^N Z_t(i)$, and $\alpha \in [0, 1]$.

The case $\alpha = 0$ corresponds to N independent copies of the classical Pólya's urn described above. Thus, for $1 \leq j \leq N$, each proportion $Z_t(j)$ converges, as $t \rightarrow +\infty$, to i.i.d. random variables, whose distribution is Beta with parameters a, b . As soon as $\alpha > 0$, some basic properties of the Pólya's urn model is lost: in particular the sequences $(Z_t(j))_t$ are not martingales (although $(Z_t)_t$ is a martingale), and the sequences of the colors drawn are no more exchangeable. It is shown in [9], that $D_t(j) := Z_t(j) - Z_t$ converges to zero almost surely and in L^2 ; as a consequence, all the fractions $Z_t(j)$ converge a.s. to the *same limit* Z . We refer to this phenomenon as *almost sure synchronization* of the system of interacting urns. It is relevant to note that this is not a macroscopic or *thermodynamic* effect: the number of urns N is kept fixed. The "phase transition" from disorder ($\alpha = 0$) to synchronization ($\alpha > 0$) is not a consequence of the large scale of the system but of the long memory caused by the reinforcement.

In the present paper we analyze in detail the *fluctuations* of $D_t(j) = Z_t(j) - Z_t$ around zero as $t \rightarrow +\infty$. The rate of convergence to zero in L^2 -norm has been already analyzed in [9], revealing an interesting scaling for certain values of the interaction parameter α : $D_t(j)$ scales as $t^{-1/2}$ for $\frac{1}{2} < \alpha \leq 1$, as $t^{-1/2} \sqrt{\ln(t)}$ for $\alpha = \frac{1}{2}$, and as $t^{-\alpha}$ for $0 < \alpha < \frac{1}{2}$. In this paper, we obtain limit theorems for the rescaled fluctuations: for $\alpha \geq \frac{1}{2}$, they converge in distribution, as $t \rightarrow +\infty$, to a mixture of centered Gaussian distributions, whose random variance is an explicit function of the limit random variable Z ; for $0 < \alpha < \frac{1}{2}$, the rescaled fluctuations converge *almost surely* to a nonzero real random variable. Indeed, in the case $\alpha \geq \frac{1}{2}$, we prove *stable convergence* (see e.g. [8]), which is strictly stronger than convergence in distribution (basic definitions and results on this form of convergence will be recalled later on).

We note that the scaling phenomenon in our model resembles the one observed in a single Friedman's urn (see [11, 12]). Friedman's urn model is a modification of Pólya's urn in which, after each drawing, one adds in the urn $A > 0$ balls of the drawn color, and $B > 0$ balls of the other color. In this model the proportion Z_t of red balls converges a.s. to $\frac{1}{2}$ and fluctuations around this limit exhibit an analogous scaling depending whether the parameter $\alpha := 1 - \frac{A-B}{A+B}$ is smaller or equal or bigger than $1/2$. (However, we point out that the limit distributions obtained by Friedman for $\alpha \geq 1/2$ are simple Gaussian distributions and not mixtures of Gaussian distributions.) The analogy can be explained by a similar equation for the conditional probability of drawing a red ball at time $t + 1$. Indeed, our arguments for the proofs in this paper could be used to strengthen some of the results in [11]. We will not elaborate further on this point.

We also mention that central limit theorems for a single randomly reinforced Pólya's urn has been recently given in [2, 4, 5, 7]. Finally, we remark that models of interacting urns have been considered by various authors (see e.g. [27] for a general survey on random processes with reinforcement). However, they are different from the one studied in this paper. In particular, in [21], the authors introduced a model which describes a system of interacting agents, modeled by urns arranged on a lattice, subject to perturbations and occasionally break down. Indeed, each urn contains b black and one white balls; at each time, a ball is drawn from a certain urn and, if it is white, then a new white ball is added in the urn, while, if it is black, then the urn comes back to the initial composition and, for each white ball previously present in the urn, a similar attempt to add a white ball is made on a randomly chosen nearest neighbour urn. In [6], the authors also consider a system of interacting urns on a lattice, representing agents subject to defaults. The reinforcement matrix is not only a function of

time (time contagion), but also of the behavior of the neighboring urns (spatial contagion) and of a random component. In [24], a countable collection of interacting urns is considered in which, at each time, a ball is sampled from each urn and a random number of new balls of the same color of the extracted one is introduced in the urn together with the drawn ball. The distribution of the reinforcement for urn j depends on the colors extracted from the urns with index $i \neq j$ and on an independent random factor. Urns with a mean-field interaction have been considered in [18, 19], but with a reinforcement scheme different from ours: their main results are proven when the probability of drawing a ball of a certain color is proportional to the exponential of the number of balls of that color, rather than to the number of balls of that color, leading to a quite different synchronization picture.

The paper is organized as follows. In Section 2 we formally introduce the model and recall the needed facts concerning stable convergence. In Section 3 we give and discuss the statement of our main results, whose proofs are postponed to Section 4. The paper is enriched with an appendix which contains some useful auxiliary results.

2 Setting, notation and preliminaries

We consider the interacting system introduced in [9]. More precisely, we have the following system of $N > 1$ (with N finite) Pólya's urns on a probability space (Ω, \mathcal{A}, P) , with “mean field interaction”:

- at time $t = 0$, each urn contains $a \geq 1$ red balls and $b \geq 1$ black balls;
- at time $t + 1$, a new ball is added in each urn as follows: given the proportion $Z_t(i)$, for $1 \leq i \leq N$, of red balls in each urn at time t , the ball added in urn j is, independently of what happens in all the urns with $i \neq j$, red (otherwise it is black) with conditional probability

$$\alpha Z_t + (1 - \alpha) Z_t(j) \quad (1)$$

where $Z_t = \frac{1}{N} \sum_{i=1}^N Z_t(i)$ is the total proportion of red balls in the system at time t and α is a parameter in $[0, 1]$ which tunes the interaction among the urns ($\alpha = 0$ corresponds to N independent Pólya's urns and $\alpha = 1$ corresponds to the admissible maximum level of interaction).

For all the sequel, we set $m = a + b$ and $D_t(j) = (Z_t(j) - Z_t)$ (for simplicity, sometimes we omit the index j , i.e. we set $D_t = D_t(j)$, when j is fixed) and, for each $i = 1, \dots, N$, we denote by $I_{t+1}(i)$ the indicator function of the event {red ball for urn i at time $t + 1$ }¹ and we define the (increasing) filtration $\mathcal{F} = (\mathcal{F}_t)_t$ as

$$\mathcal{F}_0 = \{\Omega, \emptyset\} \quad \text{and} \quad \mathcal{F}_t = \sigma(I_k(i) : i = 1, \dots, N, 1 \leq k \leq t).$$

It is easy to verify that (Z_t) is an \mathcal{F} -martingale which converges almost surely to a random variable Z . Furthermore, in [9], authors verified that each $(Z_t(j))$ is an \mathcal{F} -quasi-martingale (a martingale when $\alpha = 0$) and, for $\alpha > 0$, they proved the almost sure synchronization, i.e.

$$Z_t(j) \xrightarrow{a.s.} Z \quad \forall j \in \{1, \dots, N\}.$$

These last two properties are a consequence of the relation

$$E[D_t(j)^2] = E[(Z_t(j) - Z_t)^2] \sim d(\alpha) \begin{cases} t^{-2\alpha} & \text{when } 0 < \alpha < 1/2 \\ t^{-1} \ln(t) & \text{when } \alpha = 1/2 \\ t^{-1} & \text{when } 1/2 < \alpha \leq 1 \end{cases} \quad (2)$$

which holds true for each fixed urn j and $t \rightarrow +\infty$ ².

¹It corresponds to the random variable $Y_t(i)$ in [9].

²The symbol $d(\alpha)$ denotes a constant in $(0, +\infty)$ and we use the notation $a_t \sim b_t$ when $\lim_{t \rightarrow +\infty} a_t/b_t = 1$.

We finally recall some simple relations for a fixed j , that will be useful in the following proofs:

$$Z_{t+1}(j) - Z_t(j) = \frac{I_{t+1}(j) - Z_t(j)}{m + t + 1}, \quad Z_{t+1} - Z_t = \frac{(\sum_{i=1}^N I_{t+1}(i)/N) - Z_t}{m + t + 1},$$

$$E[I_{t+1}(j) | \mathcal{F}_t] = \alpha Z_t + (1 - \alpha) Z_t(j), \quad E\left[\sum_{i=1}^N I_{t+1}(i)/N | \mathcal{F}_t\right] = Z_t,$$

and

$$E[Z_{t+1}(j) | \mathcal{F}_t] - Z_t(j) = \frac{\alpha}{m + t + 1}(Z_t - Z_t(j)) = \frac{-\alpha D_t}{m + t + 1}.$$

We conclude this section with a brief review on stable convergence.

2.1 Stable convergence and its generalizations

Stable convergence has been introduced by Rényi in [28] and subsequently investigated by various authors, e.g. [1, 8, 10, 16, 26]. It is a strong form of convergence in distribution, in the sense that it is intermediate between the simple convergence in distribution and the convergence in probability. We recall here some basic definitions. For more details, we refer the reader to [8, 15] and the references therein.

Let (Ω, \mathcal{A}, P) be a probability space, and let S be a Polish space, endowed with its Borel σ -field. A *kernel* on S , or a random probability measure on S , is a collection $K = \{K(\omega) : \omega \in \Omega\}$ of probability measures on the Borel σ -field of S such that, for each bounded Borel real function f on S , the map

$$\omega \mapsto K(f)(\omega) = \int f(x) K(\omega)(dx)$$

is \mathcal{A} -measurable.

On (Ω, \mathcal{A}, P) , let (Y_t) be a sequence of S -valued random variables and let K be a kernel on S . Then we say that Y_t converges *stably* to K , and we write $Y_t \xrightarrow{\text{stably}} K$, if

$$P(Y_t \in \cdot | H) \xrightarrow{\text{weakly}} E[K(\cdot) | H] \quad \text{for all } H \in \mathcal{A} \text{ with } P(H) > 0.$$

Clearly, if $Y_t \xrightarrow{\text{stably}} K$, then Y_t converges in distribution to the probability distribution $E[K(\cdot)]$. Moreover the convergence in probability of Y_t to a random variable Y is equivalent to the stable convergence of Y_t to a special kernel, which is the Dirac kernel $K = \delta_Y$.

We next mention two generalizations of the notion of stable convergence: a strong form of stable convergence, introduced and studied in [8], and the almost sure conditional convergence, introduced and studied in [7], that will be used later on.

For each t , let \mathcal{F}_t be a sub- σ -field of \mathcal{A} . We say that Y_t converges to K *stably in the strong sense*, with respect to the sequence $\mathcal{F} = (\mathcal{F}_t)$ (called conditioning system), if

$$E[f(Y_t) | \mathcal{F}_t] \xrightarrow{P} K(f)$$

for each bounded continuous real function f on S .

A strengthening of the stable convergence in the strong sense is the following. If K_t denotes a version of the conditional distribution of Y_t given \mathcal{F}_t , we say that Y_t converges to K in the sense of the *almost sure conditional convergence*, with respect to \mathcal{F} , if, for almost every ω in Ω , the probability measure $K_t(\omega)$ converges weakly to $K(\omega)$ and so

$$E[f(Y_t) | \mathcal{F}_t] \xrightarrow{a.s.} K(f)$$

for each bounded continuous real function f on S .

3 Results

This section is devoted to the statement of our main results, whose proofs are postponed to Section 4. The first result is a Central Limit Theorem for the total fraction Z_t of red balls in the whole system. Note that the fluctuations of Z_t around Z scale as $t^{-1/2}$ for all values of α .

Theorem 3.1. *We have*

$$\sqrt{t}(Z_t - Z) \xrightarrow{\text{stably}} \mathcal{N}\left(0, \frac{1}{N}(Z - Z^2)\right)$$

The above sequence also converges in the sense of the almost sure conditional convergence with respect to the filtration \mathcal{F} .

As observed in [2], an advantage of having the almost sure conditional convergence is the fact that it allows to prove that the distribution of the limit random variable Z has no point masses in $(0, 1)$. As a consequence of this, we get that the limit Gaussian kernel in the above theorem is not degenerate. Indeed, the following result holds true.

Theorem 3.2. *We have $P(Z = 0) + P(Z = 1) < 1$ and $P(Z = z) = 0$ for each $z \in (0, 1)$.*

Theorems 3.3 and 3.5 below are the main results of this paper. They are concerned with the fluctuations of $D_t(j) = Z_t(j) - Z_t$.

Theorem 3.3. *For $1/2 < \alpha \leq 1$, we have*

$$\sqrt{t}(Z_t(j) - Z_t) \xrightarrow{\text{stably}} \mathcal{N}\left(0, \frac{(1 - \frac{1}{N})}{(2\alpha - 1)}(Z - Z^2)\right).$$

For $\alpha = 1/2$, we have

$$\frac{\sqrt{t}}{\sqrt{\ln(t)}}(Z_t(j) - Z_t) \xrightarrow{\text{stably}} \mathcal{N}\left(0, \left(1 - \frac{1}{N}\right)(Z - Z^2)\right).$$

By Theorem 3.2, these limit Gaussian kernels are not degenerate.

Note that, using the Cramér-Wold device (i.e. working with a linear combination of the vector components), with a proof analogous to the one of Theorem 3.3, we can obtain the multivariate version of the above convergences. Indeed, for instance, in the case $1/2 < \alpha \leq 1$, we can get

$$\sqrt{t}\mathbf{D}_t := \sqrt{t}[Z_t(1) - Z_t, \dots, Z_t(N) - Z_t] \xrightarrow{\text{stably}} \mathcal{N}\left(0, \frac{\mathbf{V}}{(2\alpha - 1)}\right)$$

where $\mathbf{V}_{j,j} = (1 - 1/N)(Z - Z^2)$ and $\mathbf{V}_{i,j} = -(Z - Z^2)/N$ for $i \neq j$.

Theorems 3.1 and 3.3 state, in particular, that $Z_t(j) - Z_t$ and $Z_t - Z$ have the same scaling when $\alpha > 1/2$. Using a result in [4] (see also Proposition A.5 in the Appendix), these two theorems combine yielding the following statement.

Theorem 3.4. *For $1/2 < \alpha \leq 1$, we have*

$$\sqrt{t}(Z_t(j) - Z) \xrightarrow{\text{stably}} \mathcal{N}\left(0, \left[\frac{1}{N} + \frac{(1 - \frac{1}{N})}{(2\alpha - 1)}\right](Z - Z^2)\right).$$

For $\alpha = 1/2$, we have

$$\frac{\sqrt{t}}{\sqrt{\ln(t)}}(Z_t(j) - Z) \xrightarrow{\text{stably}} \mathcal{N}\left(0, \left(1 - \frac{1}{N}\right)(Z - Z^2)\right).$$

For $\alpha < \frac{1}{2}$, synchronization is slower, as $D_t(j) = Z_t(j) - Z_t$ scales like $t^{-\alpha}$, as the following Theorem establishes.

Theorem 3.5. *For $0 < \alpha < 1/2$, we have*

$$\tilde{D}_t(j) := t^\alpha(Z_t(j) - Z_t) \xrightarrow{\text{a.s.}/L^1} \tilde{D}$$

for some real random variable \tilde{D} with $P(\tilde{D} \neq 0) > 0$.

Again, if we combine the above result with Theorem 3.1, observing that

$$t^\alpha(Z_t(j) - Z) = t^\alpha(Z_t(j) - Z_t) + t^{\alpha-1/2}\sqrt{t}(Z_t - Z),$$

we obtain the following corollary.

Corollary 3.6. *For $0 < \alpha < 1/2$, we have*

$$t^\alpha(Z_t(j) - Z) \xrightarrow{P} \tilde{D}.$$

We conclude this section with a comment on the possible statistical applications of the shown results.

Remark 3.7. It is worthwhile to note that, since $U_t = (Z_t - Z_t^2)$ is a strongly consistent estimator of $U = (Z - Z^2)$, from Theorem 3.1 and 3.3 we obtain the convergences in distribution to the standard normal distribution $\mathcal{N}(0, 1)$ of the following sequences:

- $\sqrt{N}t(Z_t - Z)/\sqrt{U_t}$ for any α ,
- $(1 - 1/N)^{-1/2}\sqrt{(2\alpha - 1)}\sqrt{t}(Z_t(j) - Z_t)/\sqrt{U_t}$ for $\alpha \in (1/2, 1]$,
- $(1 - 1/N)^{-1/2}\sqrt{t \ln(t)^{-1/2}}(Z_t(j) - Z_t)/\sqrt{U_t}$ for $\alpha = 1/2$.

The first of the above convergences can be used in order to provide an asymptotic confidence interval for the limit random variable Z (see [5]). The other two can be useful in order to construct statistical tests and asymptotic confidence intervals for the parameter α .

4 Proofs

4.1 Proof of Theorem 3.1

We will prove the almost sure conditional convergence (and so the stable convergence) using Theorem 2.2 in [7] (see also Prop. 1 in [4]). To this purpose, we observe that $(Z_t)_t$ is an \mathcal{F} -martingale which converges to a random variable Z a.s. and in mean and which satisfies the following two conditions:

- 1) $E \left[\sup_k \sqrt{k} |Z_{k+1} - Z_k| \right] < +\infty$;
- 2) $t \sum_{k \geq t} (Z_{k+1} - Z_k)^2 \xrightarrow{a.s.} \frac{1}{N} (Z - Z^2)$.

Indeed, the first condition immediatly follows from

$$|Z_{k+1} - Z_k| = \frac{1}{m+k+1} \left| \frac{\sum_{i=1}^N I_{k+1}(i)}{N} - Z_k \right| = O(k^{-1}).$$

Regarding the second condition, we observe that

$$t \sum_{k \geq t} (Z_{k+1} - Z_k)^2 = t \sum_{k \geq t} \frac{1}{(m+k+1)^2} \left(\frac{\sum_{i=1}^N I_{k+1}(i)}{N} - Z_k \right)^2$$

and so the desired convergence follows by Lemma A.1 (with $a_k = 1$, $b_k = k$, $Y_k = k^2 \left(\frac{\sum_{i=1}^N I_{k+1}(i)}{N} - Z_k \right)^2 / (m+k+1)^2$ and $\mathcal{G}_k = \mathcal{F}_{k+1}$) since

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \frac{k^4 E \left[\left(\frac{\sum_{i=1}^N I_{k+1}(i)}{N} - Z_k \right)^4 \right]}{(m+k+1)^4} \leq \sum_{k=1}^{\infty} \frac{1}{k^2} < +\infty$$

and (taking into account the \mathcal{F}_k -conditional independence of the indicator functions $I_{k+1}(i)$ for $i = 1, \dots, N$)

$$\begin{aligned}
& \frac{k^2}{(m+k+1)^2} E \left[\left(\frac{\sum_{i=1}^N I_{k+1}(i)}{N} - Z_k \right)^2 \mid \mathcal{F}_k \right] \\
&= \frac{k^2}{(m+k+1)^2} \text{Var} \left[\frac{\sum_{i=1}^N I_{k+1}(i)}{N} \mid \mathcal{F}_k \right] \\
&= \frac{k^2}{(m+k+1)^2} \frac{1}{N^2} \sum_{i=1}^N \text{Var}[I_{k+1}(i) \mid \mathcal{F}_k] \\
&= \frac{k^2}{(m+k+1)^2} \frac{1}{N^2} \sum_{i=1}^N [\alpha Z_k + (1-\alpha)Z_k(i) - (\alpha Z_k + (1-\alpha)Z_k(i))^2] \\
&\xrightarrow{a.s.} \frac{1}{N} (Z - Z^2).
\end{aligned}$$

4.2 Proof of Theorem 3.2

The first part of Theorem 3.2 immediately follows from the relation $E[Z^2] < E[Z]$, that is $E[Z(1-Z)] > 0$, (see Lemma 1 in [9]).

The second part of Theorem 3.2 is a consequence of the almost sure conditional convergence stated in Theorem 3.1. Indeed, if we denote by K_t a version of the conditional distribution of $\sqrt{t}(Z_t - Z)$ given \mathcal{F}_t , then there exists an event A such that $P(A) = 1$ and, for each $\omega \in A$,

$$\lim_t Z_t(\omega) = Z(\omega) \quad \text{and} \quad K_t(\omega) \xrightarrow{weakly} \mathcal{N} \left(0, \frac{1}{N} (Z(\omega) - Z^2(\omega)) \right).$$

Assume now, by absurd, that there exists $z \in (0, 1)$ with $P(Z = z) > 0$, and set $A' = A \cap \{Z = z\}$ and define B_t as the \mathcal{F}_t -measurable random set $\{\sqrt{t}(Z_t - z)\}$. Then $P(A') > 0$ and, since $E[I_{\{Z=z\}} \mid \mathcal{F}_t]$ converges almost surely to $I_{\{Z=z\}}$, there exists an event A'' such that $P(A'') > 0$, $A'' \subseteq A'$ and, for each $\omega \in A''$,

$$K_t(\omega)(B_t(\omega)) = E \left[I_{\{\sqrt{t}(Z_t - z)\}} \left(\sqrt{t}(Z_t - Z) \right) \mid \mathcal{F}_t \right] (\omega) = E \left[I_{\{Z=z\}} \mid \mathcal{F}_t \right] (\omega) \longrightarrow I_{\{Z=z\}}(\omega) = 1.$$

On the other hand, we observe that $N^{-1}(Z(\omega) - Z^2(\omega))$ is not zero when $\omega \in A'$. Hence, if D is the discrepancy metric defined by

$$D[\mu, \nu] = \sup_{B \in \{\text{closed balls of } \mathbb{R}\}} |\mu(B) - \nu(B)|,$$

which metrizes the weak convergence of a sequence of probability distributions on \mathbb{R} in the case when the limit distribution is absolutely continuous with respect to the Lebesgue measure on \mathbb{R} (see [14]), then, for each $\omega \in A'$, we have

$$\begin{aligned}
K_t(\omega)(B_t(\omega)) &= \left| K_t(\omega)(B_t(\omega)) - \mathcal{N} \left(0, \frac{1}{N} (Z(\omega) - Z^2(\omega)) \right) (B_t(\omega)) \right| \\
&\leq D \left[K_t(\omega), \mathcal{N} \left(0, \frac{1}{N} (Z(\omega) - Z^2(\omega)) \right) \right] \longrightarrow 0.
\end{aligned}$$

This contradicts the previous fact and the proof is concluded.

4.3 Proof of Theorems 3.3 and 3.4

Proof of Theorem 3.3. Let us set

$$\begin{aligned}
L_0 &= D_0 = 0 \\
L_t &= D_t - \sum_{k=0}^{t-1} (E[D_{k+1}|\mathcal{F}_k] - D_k) \\
&= D_t + \alpha \sum_{k=0}^{t-1} \frac{D_k}{m+k+1} \\
&= D_t + \alpha \sum_{k=1}^{t-1} \frac{D_k}{m+k+1} \quad \text{per } t \geq 1.
\end{aligned}$$

Then (L_t) is an \mathcal{F} -martingale by construction and, for each $t \geq 1$, we can write

$$D_{t+1} = \left(1 - \frac{\alpha}{m+t+1}\right) D_t + \Delta L_{t+1},$$

where $\Delta L_{t+1} = L_{t+1} - L_t$. Iterating the above relation and using the notation in Lemma A.2, we obtain

$$D_{t+1} = c_{1,t} D_1 + \sum_{k=1}^t c_{k+1,t} \Delta L_{k+1}.$$

We firstly consider the case $\alpha > 1/2$. By Lemma A.2, we have $\sqrt{t}c_{1,t} \sim ct^{-(\alpha-1/2)} \rightarrow 0$ since $\alpha > 1/2$. Therefore, it is enough to prove the convergence

$$\sqrt{t} \sum_{k=1}^t c_{k+1,t} \Delta L_{k+1} \xrightarrow{\text{stably}} \mathcal{N}\left(0, \frac{V}{2\alpha-1}\right),$$

where $V = (1 - 1/N)(Z - Z^2)$. To this purpose, let us define

$$Y_{t,k} = \sqrt{t} c_{k+1,t} \Delta L_{k+1} \quad \text{and} \quad \mathcal{G}_{t,k} = \mathcal{F}_{k+1}.$$

Thus, $\{Y_{t,k}, \mathcal{G}_{t,k} : 1 \leq k \leq t\}$ is a square-integrable martingale difference array. Indeed, we have

$$E[Y_{t,k}^2] < +\infty \quad \text{and} \quad E[Y_{t,k+1}|\mathcal{G}_{t,k}] = \sqrt{t} c_{k+2,t} E[\Delta L_{k+2}|\mathcal{F}_{k+1}] = 0.$$

Applying Theorem A.4, the convergence

$$\sqrt{t} \sum_{k=1}^t c_{k+1,t} \Delta L_{k+1} = \sum_{k=1}^t Y_{t,k} \xrightarrow{\text{stably}} \mathcal{N}\left(0, \frac{V}{2\alpha-1}\right)$$

is ensured if the following conditions are satisfied:

- 1) $\max_{1 \leq k \leq t} |Y_{t,k}| \xrightarrow{P} 0$;
- 2) $E[\max_{1 \leq k \leq t} Y_{t,k}^2]$ is bounded in t ;
- 3) $\sum_{k=1}^t Y_{t,k}^2 \xrightarrow{P} V/(2\alpha-1)$.

Hence, in order to conclude the proof, we now verify the above three conditions.

PROOF OF CONDITION 1). We observe that

$$\begin{aligned}
\Delta L_{k+1} &= D_{k+1} - D_k + \alpha \frac{D_k}{m+k+1} \\
&= (Z_{k+1}(j) - Z_k(j)) - (Z_{k+1} - Z_k) + \alpha \frac{D_k}{m+k+1} \\
&= \frac{I_{k+1}(j) - Z_k(j)}{m+k+1} - \frac{(\sum_{i=1}^N I_{k+1}(i)/N) - Z_k}{m+k+1} + \alpha \frac{D_k}{m+k+1}
\end{aligned}$$

and so $|\Delta L_{k+1}| = O(k^{-1})$. Therefore, condition 1) easily follows since³

$$(\max_{1 \leq k \leq t} |Y_{t,k}|)^3 \leq \sum_{k=1}^{t-1} |Y_{t,k}|^3 + |Y_{t,t}|^3 \simeq \frac{1}{t^{3\alpha-3/2}} \sum_{k=1}^{t-1} \frac{k^3 O(k^{-3})}{k^{1-(3\alpha-2)}} + \frac{t^3 O(t^{-3})}{t^{3/2}} \longrightarrow 0.$$

PROOF OF CONDITION 2). We have

$$\begin{aligned} E[\max_{1 \leq k \leq t} Y_{t,k}^2] &\leq \sum_{k=1}^t E[Y_{t,k}^2] = t \sum_{k=1}^t c_{k+1,t}^2 E[(\Delta L_{k+1})^2] \\ &= t \sum_{k=1}^{t-1} c_{k+1,t}^2 E[(\Delta L_{k+1})^2] + t E[(\Delta L_{t+1})^2] \\ &\simeq \frac{1}{t^{2\alpha-1}} \sum_{k=1}^{t-1} \frac{k^2 O(k^{-2})}{k^{1-(2\alpha-1)}} + \frac{t^2 O(t^{-2})}{t}. \end{aligned}$$

The last term is bounded in t since

$$\sum_{k=1}^{t-1} \frac{1}{k^{1-(2\alpha-1)}} \sim \frac{t^{2\alpha-1}}{2\alpha-1}.$$

PROOF OF CONDITION 3). We observe that

$$\sum_{k=1}^t Y_{t,k}^2 = t \sum_{k=1}^t c_{k+1,t}^2 (\Delta L_{k+1})^2 \simeq \frac{1}{t^{2\alpha-1}} \sum_{k=1}^{t-1} \frac{(k \Delta L_{k+1})^2}{k^{1-(2\alpha-1)}} + \frac{(t \Delta L_{t+1})^2}{t}.$$

Moreover, we have

$$\begin{aligned} (\Delta L_{k+1})^2 &= (Z_{k+1}(j) - Z_k(j))^2 + (Z_{k+1} - Z_k)^2 + \alpha^2 \frac{D_k^2}{(m+k+1)^2} \\ &\quad - 2(Z_{k+1}(j) - Z_k(j))(Z_{k+1} - Z_k) + 2\alpha \frac{(Z_{k+1}(j) - Z_k(j))D_k}{m+k+1} - 2\alpha \frac{(Z_{k+1} - Z_k)D_k}{m+k+1} \\ &= (Z_{k+1}(j) - Z_k(j))^2 + (Z_{k+1} - Z_k)^2 - 2(Z_{k+1}(j) - Z_k(j))(Z_{k+1} - Z_k) \\ &\quad + \alpha^2 \frac{D_k^2}{(m+k+1)^2} + 2\alpha \frac{(I_{k+1}(j) - Z_k(j))D_k}{(m+k+1)^2} - 2\alpha \frac{(\sum_{i=1}^N I_{k+1}(i)/N - Z_k)D_k}{(m+k+1)^2}. \end{aligned}$$

Since $D_k \xrightarrow{a.s.} 0$, it is easy to get that each of the three terms

$$\begin{aligned} &\frac{1}{t^{2\alpha-1}} \sum_{k=1}^{t-1} \frac{k^2 D_k^2}{(m+k+1)^2 k^{1-(2\alpha-1)}}, \\ &\frac{1}{t^{2\alpha-1}} \sum_{k=1}^{t-1} \frac{k^2 (I_{k+1}(j) - Z_k(j))D_k}{(m+k+1)^2 k^{1-(2\alpha-1)}}, \\ &\frac{1}{t^{2\alpha-1}} \sum_{k=1}^{t-1} \frac{k^2 (\sum_{i=1}^N I_{k+1}(i)/N - Z_k)D_k}{(m+k+1)^2 k^{1-(2\alpha-1)}} \end{aligned}$$

converges almost surely to zero. Hence, setting $a_k = k^{2-2\alpha}$, $b_k = k^{2\alpha-1}$, $\mathcal{G}_k = \mathcal{F}_{k+1}$ and

$$Y_k = k^2[(Z_{k+1}(j) - Z_k(j))^2 + (Z_{k+1} - Z_k)^2 - 2(Z_{k+1}(j) - Z_k(j))(Z_{k+1} - Z_k)],$$

by Lemma A.1, condition 3) is satisfied if

$$\sum_{k=1}^{\infty} \frac{E[Y_k^2]}{k^2} < +\infty \quad \text{and} \quad E[Y_k | \mathcal{F}_k] \xrightarrow{a.s.} V.$$

³We use the notation $a_t \simeq b_t$ when $\lim_t a_t = \lim_t b_t$.

The first condition is trivially satisfied since $Y_k^2 = k^4 O(k^{-4})$. As concerns the second one, we have already verified in the proof of Theorem 3.1 that

$$E[k^2(Z_{k+1} - Z_k)^2 | \mathcal{F}_k] = \frac{k^2}{(m+k+1)^2} E\left[\left(\frac{\sum_{i=1}^N I_{k+1}(i)}{N} - Z_k\right)^2 | \mathcal{F}_k\right] \xrightarrow{a.s.} \frac{1}{N} (Z - Z^2).$$

Further, we can check that

$$E[k^2(Z_{k+1}(j) - Z_k(j))^2 | \mathcal{F}_k] = \frac{k^2}{(m+k+1)^2} E[(I_{k+1}(j) - Z_k(j))^2 | \mathcal{F}_k] \xrightarrow{a.s.} (Z - Z^2).$$

Finally, we observe that

$$\begin{aligned} E[k^2(Z_{k+1}(j) - Z_k(j))(Z_{k+1} - Z_k) | \mathcal{F}_k] &= \\ \frac{k^2}{(m+k+1)^2} E\left[(I_{k+1}(j) - Z_k(j)) \left(\frac{\sum_{i=1}^N I_{k+1}(i)}{N} - Z_k\right) | \mathcal{F}_k\right] &= \\ \frac{k^2}{(m+k+1)^2} E\left[\frac{I_{k+1}(j)}{N} + \frac{\sum_{i=1, i \neq j}^N I_{k+1}(i) I_{k+1}(j)}{N} - Z_k I_{k+1}(j) - Z_k(j) \frac{\sum_{i=1}^N I_{k+1}(i)}{N} + Z_k Z_k(j) | \mathcal{F}_k\right] &= \\ \xrightarrow{a.s.} Z/N + Z^2 - Z^2/N - Z^2 - Z^2 + Z^2 = Z/N - Z^2/N = \frac{1}{N} (Z - Z^2). \end{aligned}$$

The proof of the case $\alpha > 1/2$ is so concluded.

Finally, the proof of the case $\alpha = 1/2$ is essentially the same. Indeed, we have $(\sqrt{t}/\sqrt{\ln(t)})c_{1,t} \sim c(\ln(t))^{-1/2} \rightarrow 0$ and so we can continue the argument as above by setting

$$Y_{t,k} = \frac{\sqrt{t}}{\sqrt{\ln(t)}} c_{k+1,t} \Delta L_{k+1}.$$

For this purpose, we observe that

$$(\max_{1 \leq k \leq t} |Y_{t,k}|)^3 \leq \sum_{k=1}^{t-1} |Y_{t,k}|^3 + |Y_{t,t}|^3 \simeq \frac{1}{\ln(t)\sqrt{\ln(t)}} \sum_{k=1}^{t-1} \frac{k^3 O(k^{-3})}{k^{1+1/2}} + \frac{t^3 O(t^{-3})}{\ln(t)\sqrt{\ln(t)} t^{3/2}} \rightarrow 0$$

and

$$E[\max_{1 \leq k \leq t} Y_{t,k}^2] \leq \frac{1}{\ln(t)} \sum_{k=1}^{t-1} \frac{k^2 O(k^{-2})}{k} + \frac{t^2 O(t^{-2})}{t \ln(t)}$$

with

$$\sum_{k=1}^{t-1} \frac{1}{k} \sim \ln(t).$$

Finally, we apply Lemma A.1 with $a_k = k$, $b_k = \ln(k)$ and $\mathcal{G}_k = \mathcal{F}_{k+1}$, in order to prove that

$$\sum_{k=1}^t Y_{t,k}^2 = \frac{t}{\ln(t)} \sum_{k=1}^t c_{k+1,t} (\Delta L_{k+1})^2 \simeq \frac{1}{\ln(t)} \sum_{k=1}^{t-1} \frac{(k \Delta L_{k+1})^2}{k} \xrightarrow{a.s.} V.$$

Proof of Theorem 3.4.

CASE $1/2 < \alpha \leq 1$. Since we can write

$$\sqrt{t}(Z_t(j) - Z) = \sqrt{t}(Z_t(j) - Z_t) + \sqrt{t}(Z_t - Z),$$

the stated convergence follows from Theorem 3.1, Theorem 3.3 and Proposition A.5. Indeed, $\sqrt{t}(Z_t(j) - Z_t)$ is \mathcal{F}_t -measurable and it converges stably and $\sqrt{t}(Z_t - Z)$ is $\bigvee_t \mathcal{F}_t$ -measurable and it converges in the sense of the almost sure conditional convergence (and so

in the sense of the strong stable convergence) with respect to \mathcal{F} .

CASE $\alpha = 1/2$. Since we can write

$$\frac{\sqrt{t}}{\sqrt{\ln(t)}}(Z_t(j) - Z) = \frac{\sqrt{t}}{\sqrt{\ln(t)}}(Z_t(j) - Z_t) + \frac{\sqrt{t}(Z_t - Z)}{\sqrt{\ln(t)}},$$

the stated convergence follows from Theorem 3.1 and Theorem 3.3. Indeed, the first term converges stably to the desired Gaussian kernel and the second one converges in probability to zero.

4.4 Proof of Theorem 3.5

First we prove a general property. Let us fix $j \in \{1, \dots, N\}$ and set

$$\tilde{D}_t = t^\alpha D_t = t^\alpha (Z_t(j) - Z_t).$$

Lemma 4.1. *For any $\alpha \in [0, 1]$, the sequence $(\tilde{D}_t)_t$ is an \mathcal{F} -quasi-martingale (an \mathcal{F} -martingale when $\alpha = 0$).*

Proof. The statement is trivial when $\alpha = 0$ (since we have N independent Pólya's urns). Therefore, let us assume $\alpha > 0$. We observe that

$$\begin{aligned} E[\tilde{D}_{t+1} | \mathcal{F}_t] - \tilde{D}_t &= \left[\left(1 + \frac{1}{t}\right)^\alpha - 1 \right] \tilde{D}_t - \left(1 + \frac{1}{t}\right)^\alpha \frac{\alpha}{m+t+1} \tilde{D}_t \\ &= \left[\frac{\alpha}{t} + O(1/t^2) \right] \tilde{D}_t - \left[\frac{\alpha}{m+t+1} + O(1/t^2) \right] \tilde{D}_t \\ &= \left[\frac{m+1}{t(m+t+1)} + O(1/t^2) \right] \alpha \tilde{D}_t \\ &= O(1/t^2) \alpha \tilde{D}_t. \end{aligned}$$

Hence, the sequence $(\tilde{D}_t)_t$ is an \mathcal{F} -quasimartingale if

$$\sum_{t=1}^{+\infty} \frac{E[|D_t|]}{t^{1+(1-\alpha)}} < +\infty.$$

This last condition is obviously true when $0 < \alpha < 1$ (since $(D_t)_t$ is uniformly bounded). Moreover it is also satisfied when $\alpha = 1$ since (2) implies $E[|D_t|] = O(t^{-1/2})$.

Proof of Theorem 3.5. By Lemma 4.1, the sequence $(\tilde{D}_t)_t$ is an \mathcal{F} -quasi-martingale. Moreover, by (2), we have $\sup_t E[\tilde{D}_t^2] < +\infty$ and so it converges a.s. and in mean to some real random variable \tilde{D} .

In order to prove that $P(\tilde{D} \neq 0) > 0$, we will prove that $(\tilde{D}_t^2)_t$ is bounded in L^p for a suitable $p > 1$. Indeed, this fact implies that \tilde{D}_t^2 converges in mean to \tilde{D}^2 and so, by (2), we obtain

$$E[\tilde{D}^2] = \lim_t E[\tilde{D}_t^2] = \lim_t t^{2\alpha} E[D_t^2] > 0.$$

To this purpose, we set $p = 1 + \epsilon/2$, with $\epsilon > 0$ and $x_t = E[|D_t|^{2+\epsilon}]$. We recall that the following recursive equation holds:

$$D_{t+1} = \frac{m+t}{m+t+1} D_t + \frac{1}{m+t+1} \left[I_{t+1}(j) - \frac{\sum_{i=1}^N I_{t+1}(i)}{N} \right].$$

Then we can write

$$\begin{aligned} x_{t+1} &= \left(\frac{m+t}{m+t+1} \right)^{2+\epsilon} E[|D_t|^{2+\epsilon}] \\ &\quad + (2+\epsilon) \left(\frac{m+t}{m+t+1} \right)^{1+\epsilon} \frac{1}{m+t+1} E \left[|D_t|^{1+\epsilon} \operatorname{sgn}(D_t) \left(I_{t+1}(j) - \frac{\sum_{i=1}^N I_{t+1}(i)}{N} \right) \right] \\ &\quad + R_t \end{aligned}$$

where $R_t = O(t^{-2})$. Now, since $E \left[I_{t+1}(j) - \frac{\sum_{i=1}^N I_{t+1}(i)}{N} \middle| \mathcal{F}_t \right] = (1 - \alpha)D_t$, we have

$$\begin{aligned}
x_{t+1} &= \left(\frac{m+t}{m+t+1} \right)^{2+\epsilon} E[|D_t|^{2+\epsilon}] \\
&\quad + (2+\epsilon) \left(\frac{m+t}{m+t+1} \right)^{1+\epsilon} \frac{(1-\alpha)}{m+t+1} E[|D_t|^{1+\epsilon} \text{sgn}(D_t)D_t] + R_t \\
&= \left[\left(\frac{m+t}{m+t+1} \right)^{2+\epsilon} + (2+\epsilon)(1-\alpha) \frac{(m+t)^{1+\epsilon}}{(m+t+1)^{2+\epsilon}} \right] E[|D_t|^{2+\epsilon}] + R_t \\
&= \left(\frac{m+t}{m+t+1} \right)^{1+\epsilon} \left[1 + \frac{(1+\epsilon)}{m+t+1} - \frac{\alpha(2+\epsilon)}{m+t+1} \right] x_t + R_t \\
&= \left(1 - \frac{\alpha(2+\epsilon)}{m+t+1} \right) x_t + g(t)
\end{aligned}$$

with $g(t) = O(t^{-2})$, where, for the last equality we have used the elementary property

$$\left(\frac{m+t}{m+t+1} \right)^{1+\epsilon} = \left(1 - \frac{1}{m+t+1} \right)^{1+\epsilon} = 1 - \frac{1+\epsilon}{m+t+1} + O(1/t^2).$$

Therefore, we have that x_t satisfies the difference equation

$$x_0 = 0 \quad x_{t+1} = \left(1 - \frac{\alpha(2+\epsilon)}{m+t+1} \right) x_t + g(t).$$

Since, for $\epsilon > 0$ sufficiently small, we have $\alpha(2+\epsilon) < 1$ and as $t \rightarrow +\infty$

$$\begin{aligned}
\prod_{k=0}^{t-1} \left(1 - \frac{\alpha(2+\epsilon)}{m+k+1} \right) &= \exp \left[\sum_{k=0}^{t-1} \ln \left(1 - \frac{\alpha(2+\epsilon)}{m+k+1} \right) \right] \\
&= \exp \left[-\alpha(2+\epsilon) \sum_{k=0}^{t-1} \frac{1}{m+k+1} + r_1(m+t) \right] \\
&= (m+t)^{-\alpha(2+\epsilon)} r_2(m+t)
\end{aligned}$$

with $\lim_t r_2(m+t) \in (0, +\infty)$, by means of Lemma A.3, we finally obtain that there exists $\epsilon > 0$ such that

$$E[|D_t|^{2+\epsilon}] = O\left(\frac{1}{t^{\alpha(2+\epsilon)}} \right)$$

and so

$$\sup_t E[|\tilde{D}_t|^{2+\epsilon}] = \sup_t t^{\alpha(2+\epsilon)} E[|D_t|^{2+\epsilon}] < +\infty.$$

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A Appendix: Some auxiliary results

The following lemma slightly generalizes Lemma 2 in [4].

Lemma A.1. Let \mathcal{G} be an (increasing) filtration and (Y_k) be an \mathcal{G} -adapted sequence of real random variables such that $E[Y_k|\mathcal{G}_{k-1}] \rightarrow Y$ a.s. for some real random variable Y . Moreover, let (a_k) and (b_k) be two sequences of strictly positive real numbers such that

$$b_k \uparrow +\infty, \quad \sum_{k=1}^{\infty} \frac{E[Y_k^2]}{a_k^2 b_k^2} < +\infty.$$

Then we have:

- a) If $\frac{1}{b_t} \sum_{k=1}^t \frac{1}{a_k} \rightarrow \gamma$ for some constant γ , then $\frac{1}{b_t} \sum_{k=1}^t \frac{Y_k}{a_k} \xrightarrow{a.s.} \gamma Y$.
- b) If $b_t \sum_{k \geq t} \frac{1}{a_k b_k^2} \rightarrow \gamma$ for some constant γ , then $b_t \sum_{k \geq t} \frac{Y_k}{a_k b_k^2} \xrightarrow{a.s.} \gamma Y$.

Proof. Let us set

$$M_t = \sum_{k=1}^t \frac{Y_k - E[Y_k|\mathcal{G}_{k-1}]}{a_k b_k}.$$

Then (M_t) is a \mathcal{G} -martingale such that

$$\sup_t E[M_t^2] \leq 4 \sum_{k=1}^{\infty} \frac{E[Y_k^2]}{a_k^2 b_k^2} < +\infty.$$

Therefore (M_t) converges almost surely and, by Kronecker's lemma, we get

$$\frac{1}{b_t} \sum_{k=1}^t b_k \frac{Y_k - E[Y_k|\mathcal{G}_{k-1}]}{a_k b_k} \xrightarrow{a.s.} 0.$$

Moreover, by Abel's lemma, we have

$$b_t \sum_{k \geq t} \frac{Y_k - E[Y_k|\mathcal{G}_{k-1}]}{a_k b_k^2} \xrightarrow{a.s.} 0.$$

This is sufficient in order to conclude since, in case a), we have

$$\frac{1}{b_t} \sum_{k=1}^t \frac{Y_k}{a_k} = \frac{1}{b_t} \sum_{k=1}^t \frac{E[Y_k|\mathcal{G}_{k-1}]}{a_k} + \frac{1}{b_t} \sum_{k=1}^t b_k \frac{Y_k - E[Y_k|\mathcal{G}_{k-1}]}{a_k b_k} \xrightarrow{a.s.} \gamma Y$$

and, in case b), we have

$$b_t \sum_{k \geq t} \frac{Y_k}{a_k b_k^2} = b_t \sum_{k \geq t} \frac{E[Y_k|\mathcal{G}_{k-1}]}{a_k b_k^2} + b_t \sum_{k \geq t} \frac{Y_k - E[Y_k|\mathcal{G}_{k-1}]}{a_k b_k^2} \xrightarrow{a.s.} \gamma Y.$$

□

Finally, we have the following technical results.

Lemma A.2. Fix $\alpha \in (0, 1]$ and set

$$c_{k,t} = \begin{cases} \prod_{h=k}^t \left(1 - \frac{\alpha}{m+h+1}\right) = \frac{\prod_{h=1}^{m+t} \left(1 - \frac{\alpha}{h+1}\right)}{\prod_{h=1}^{m+k-1} \left(1 - \frac{\alpha}{h+1}\right)} & \text{for } 1 \leq k \leq t \\ 1 & \text{for } k = t+1. \end{cases}$$

Then

$$c_{1,t} \sim ct^{-\alpha} \text{ with } c \in (0, +\infty) \quad \text{and} \quad \lim_{k \rightarrow +\infty} \sup_{t \geq k} \left| \frac{c_{k,t}}{\left(\frac{k}{t}\right)^\alpha} - 1 \right| = 0.$$

Proof. It is enough to observe that, as $t \rightarrow +\infty$, we have

$$\begin{aligned} \prod_{h=1}^{m+t} \left(1 - \frac{\alpha}{h+1}\right) &= \exp \left[\sum_{h=1}^{m+t} \ln \left(1 - \frac{\alpha}{h+1}\right) \right] \\ &= \exp \left(-\alpha \sum_{h=0}^{m+t} \frac{1}{h+1} + g_1(m+t) \right) \\ &= \exp(-\alpha \ln(m+t) + g_2(m+t)) = \\ &= (m+t)^{-\alpha} g_3(m+t) \end{aligned}$$

with $\lim_{h \rightarrow +\infty} g_3(h) \in (0, +\infty)$. \square

Lemma A.3. *The solution of the difference equation*

$$x_0 = 0 \quad x_{t+1} = f(t)x_t + g(t) \quad \text{for } t \geq 0,$$

with $f(t) > 0$ for all $t \geq 0$, is given by

$$x_0 = 0 \quad x_t = \prod_{k=0}^{t-1} f(k) \sum_{i=0}^{t-1} \frac{g(i)}{\prod_{k=0}^i f(k)} \quad \text{for } t \geq 1.$$

Proof. If we set $y_0 = x_0$ and $y_t = x_t / \prod_{k=0}^{t-1} f(k)$ for $t \geq 1$, then we find that y_t satisfies the difference equation

$$y_0 = 0 \quad y_{t+1} = y_t + F(t) \quad \text{where } F(t) = \frac{g(t)}{\prod_{k=0}^t f(k)}.$$

Hence, it is easy to verify that $y_t = \sum_{i=0}^{t-1} F(i)$ for $t \geq 1$ and so the proof is concluded. \square

We next state two results without proofs.

Theorem A.4. *(Theorem 3.2 in [15])*

Let $\{S_{n,k}, \mathcal{F}_{n,k} : 1 \leq k \leq k_n, n \geq 1\}$ be a zero-mean, square-integrable martingale array with differences $Y_{n,k}$, and let η^2 be an a.s. finite random variable. Suppose that

- 1) $\max_{1 \leq k \leq k_n} |Y_{n,k}| \xrightarrow{P} 0$;
- 2) $E[\max_{1 \leq k \leq k_n} Y_{n,k}^2]$ is bounded in n ;
- 3) $\sum_{k=1}^{k_n} Y_{n,k}^2 \xrightarrow{P} \eta^2$

and the σ -fields are nested, i.e. $\mathcal{F}_{n,k} \subseteq \mathcal{F}_{n+1,k}$ for $1 \leq k \leq k_n, n \geq 1$. Then $S_{n,k_n} = \sum_{k=1}^{k_n} Y_{n,k}$ converges stably to a random variable with characteristic function $\varphi(u) = E[\exp(-\eta^2 u^2/2)]$, i.e. to the Gaussian kernel $\mathcal{N}(0, \eta^2)$.

(Here, the symbol $\mathcal{N}(0, 0)$ means the Dirac distribution ϵ_0 .)

Proposition A.5. *(Lemma 1 in [4])*

Suppose that C_n and D_n are S -valued random variables, that M and N are kernels on S , and that $\mathcal{G} = (\mathcal{G}_n)_n$ is an (increasing) filtration satisfying for all n

$$\sigma(C_n) \subseteq \mathcal{G}_n \quad \text{and} \quad \sigma(D_n) \subseteq \mathcal{G}_\infty = \sigma(\bigcup_n \mathcal{G}_n)$$

If C_n stably converges to M and D_n converges to N stably in the strong sense, with respect to \mathcal{G} , then

$$(C_n, D_n) \xrightarrow{\text{stably}} M \times N.$$

(Here, $M \times N$ is the kernel on $S \times S$ such that $(M \times N)(\omega) = M(\omega) \times N(\omega)$ for all ω .)

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